Energy Balance of a Bose Gas in a general curved space-time.

T. Matos, A. Avilez, T. Bernal and P. H. Chavanis. arXiv:1608.03945

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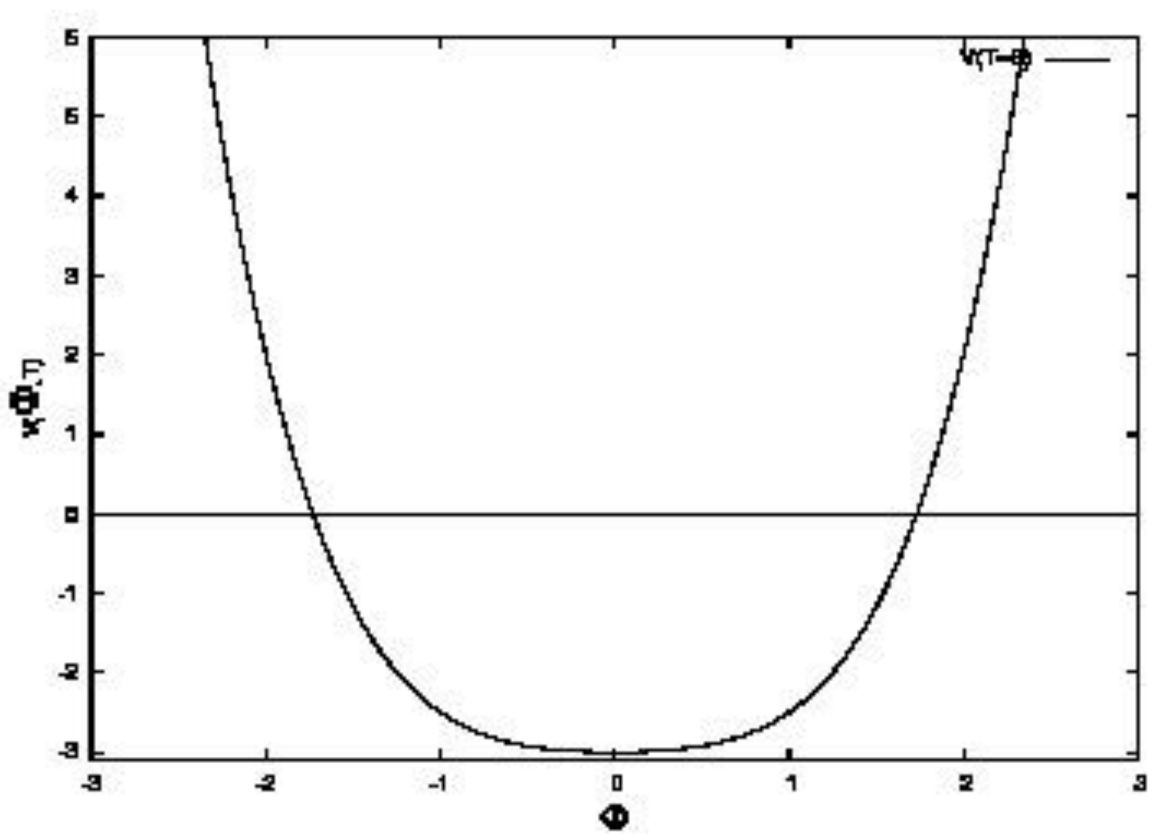
Bose Gas

Phase Transition

Canonical quantization

Klein-Gordon equation $$\Box^2\Phi - \frac{dV}{d\Phi} = 0$$

$$\ddot{\Phi} + \frac{dV}{d\Phi} =$$



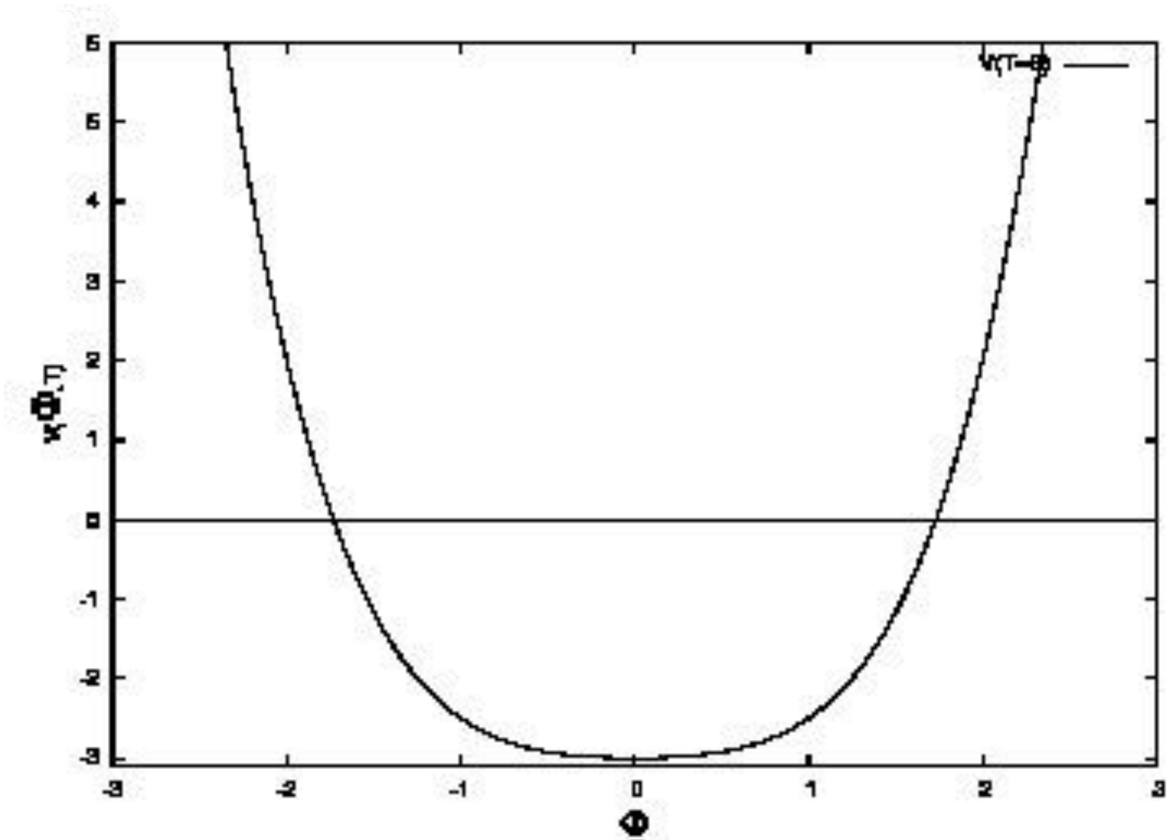
$$+ V = E$$

The Scalar Field Potential

$$V_T(\Phi) = -\frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4}\Phi^4 + \frac{\lambda}{8}T^2\Phi^2 - \frac{\pi^2}{90}T^4$$

$$\boldsymbol{T \gtrless T_c}$$

$$T_c = \frac{2m}{\sqrt{\lambda}}$$



Complex Scalar Field

A brief Review of the Scalar Field Dark Matter model. Juan Magana, Tonatiuh Matos, Victor Robles, Abril Suarez. arXiv:1201.6107

$$\mathcal{L} = \left(\nabla_\mu \Phi + \mathrm{i} e A_\mu \Phi\right)\left(\nabla^\mu \Phi^* - \mathrm{i} e A^\mu \Phi^*\right) + V(|\Phi|) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$V(\Phi) = -\hat{m}^2 \Phi\Phi^* + \frac{\hat{\lambda}}{2}(\Phi\Phi^*)^2 + \frac{\hat{\lambda}}{4} k_B^2 T^2 \Phi\Phi^* - \frac{\pi^2 k_B^4}{90\hbar^2 c^2} T^4.$$

$$\begin{aligned} \Box_{\mathrm{E}} \Phi - \frac{\mathrm{d}V}{\mathrm{d}\Phi^*} &= 0 \\ \nabla_\nu F^{\nu\mu} &= J^\mu \end{aligned}$$

$$\Box_E^2 \equiv \left(\nabla_\mu + \mathrm{i}\hat{e} A_\mu\right)\left(\nabla^\mu + \mathrm{i}\hat{e} A^\mu\right)$$

$$\Box_E^2 = (\boldsymbol{\nabla} + 2\mathrm{i}\hat{e}\boldsymbol{A}) \cdot \boldsymbol{\nabla} - \left(\frac{\partial}{\partial t} + 2\mathrm{i}\hat{e}\varphi\right)\frac{\partial}{\partial t} + \mathrm{i}\hat{e}\nabla_\mu A^\mu - \hat{e}^2 A_\mu A^\mu$$

Bose-Einstein Gas

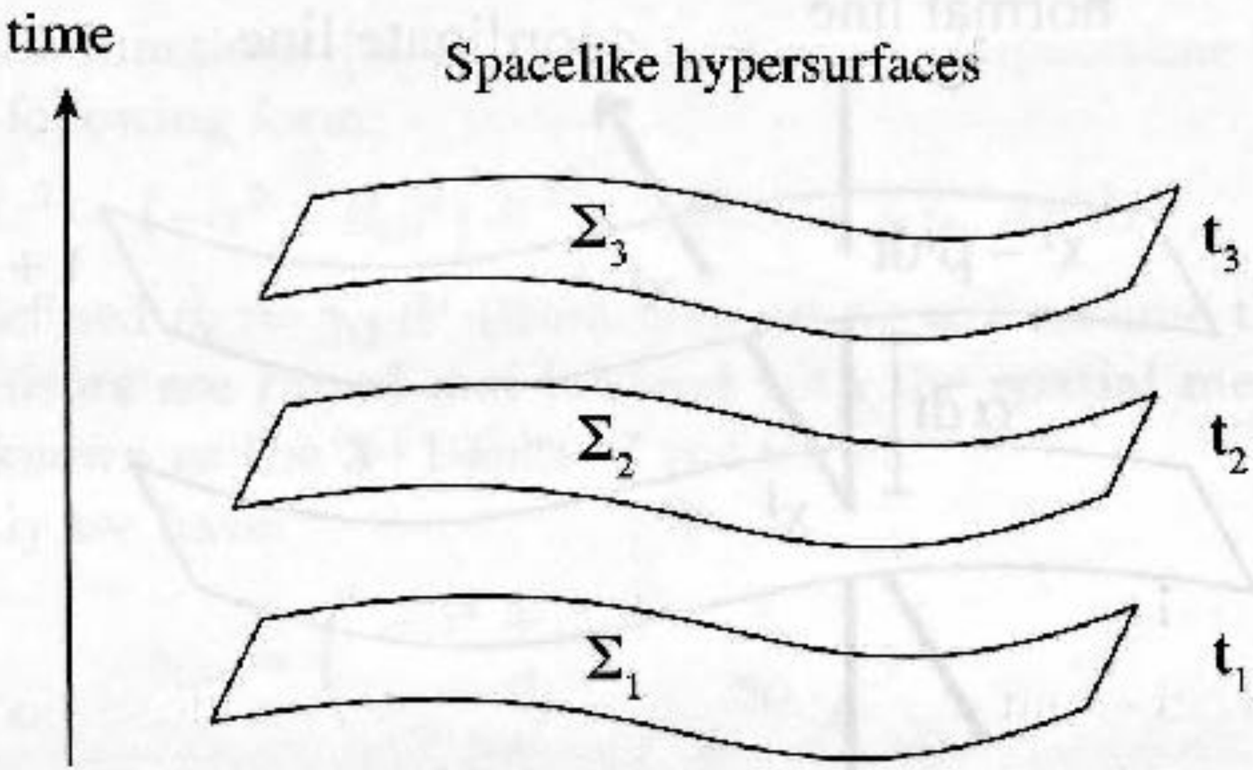
$$\begin{aligned} \Box_{\mathrm{E}}\Phi - \frac{\mathrm{d}V}{\mathrm{d}\Phi^*} &= 0 \\ \nabla_\nu F^{\nu\mu} &= J^\mu \end{aligned}$$

$$\Box_{\mathrm{E}} := \left[\nabla^\mu + i(e/\hbar c)A^\mu\right]\left[\nabla_\mu + i(e/\hbar c)A_\mu\right]$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

$$J_\mu = i\hat{e}\left[\Phi\left(\nabla_\mu - i\hat{e}A_\mu\right)\Phi^* - \Phi^*\left(\nabla_\mu + i\hat{e}A_\mu\right)\Phi\right]$$

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}$$



2.1: Foliation of spacetime into three-dimensional spacelike hypersurfaces.

$$\mathrm{d}s^2 = -\alpha^2 c^2 \mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + \beta^i c\,\mathrm{d}t\right)\left(\mathrm{d}x^j + \beta^j c\,\mathrm{d}t\right)$$

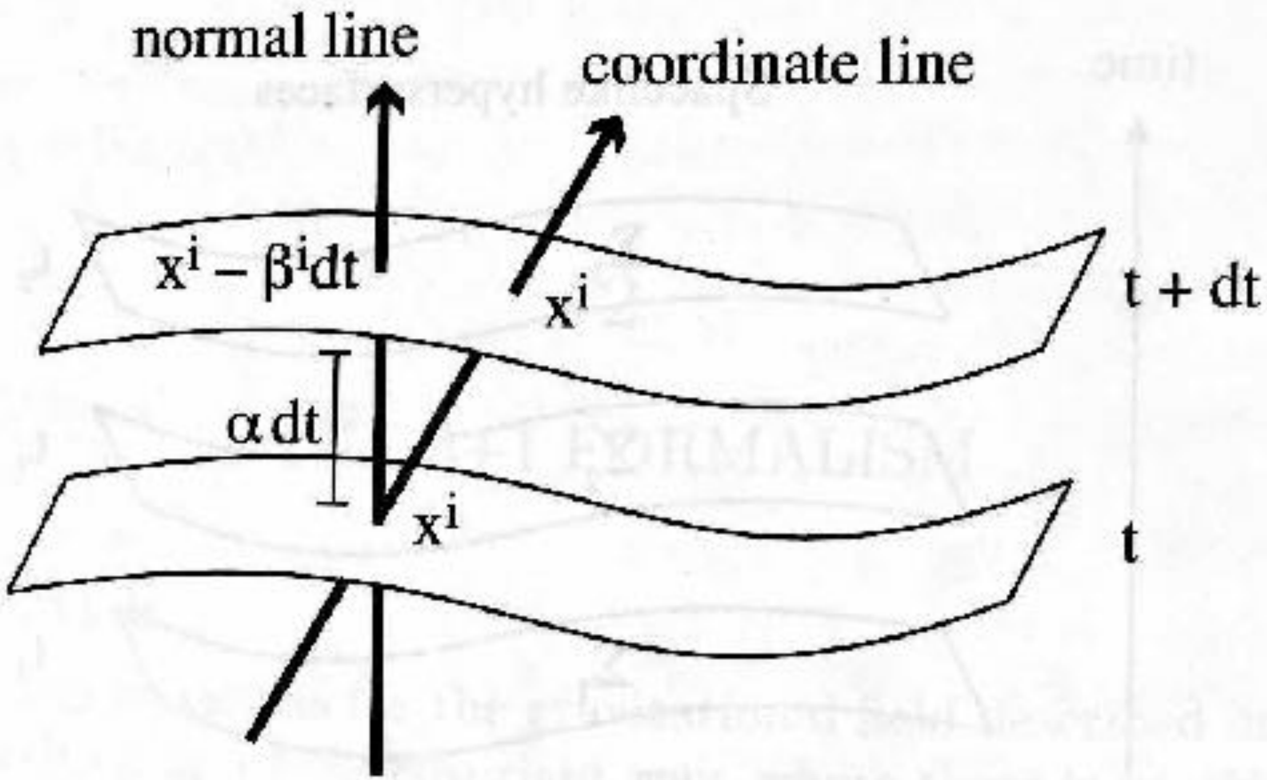


Fig. 2.2: Two adjacent spacelike hypersurfaces. The figure shows the definitions of the lapse function α and the shift vector β^i.

Generalized Gross-Pitaevskii equation

T. Matos, A. Avilez, T. Bernal and P-H. Chavanis. Energy Balance of a Bose gas in Curved Space-time. arXiv:1608.03945.

$$\Phi(t, \mathbf{x}) = \Psi(t, \mathbf{x}) \exp(-i\omega_0 t)$$

$$i\nabla^0\Psi - \frac{1}{2\omega_0}\Box_E\Psi + \frac{1}{2\omega_0}\left(\hat{m}^2 + \hat{\lambda}n\right)\Psi$$

$$+ \quad \frac{1}{2}\left(-\frac{\omega_0}{N^2} - 2\hat{e}A^0 + i\,\Box\, t\right)\Psi = 0$$

$$\Phi\Phi^* = \Psi\Psi^* = n$$

$$\nabla^0 = -\partial_0,$$
$$N = 1$$
$$\nabla^\mu = (-\dot{\ }, \nabla),$$
$$\hat{m} = \omega_0,$$
$$\Box = \nabla^2$$

Generalized Gross-Pitaevskii equation

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$$\Phi(t,\mathbf{x}) = \Psi(t,\mathbf{x})\exp(-i\omega_0 t)$$

$$-i\dot{\Psi} - \frac{1}{2\hat{m}}\nabla^2\Psi + \frac{1}{2\hat{m}}\left(\hat{m}^2 + \hat{\lambda}|\Psi|^2\right)\Psi$$

$$+ \quad \frac{1}{2}\left(-\hat{m}\right)\Psi = 0,$$

$$\Phi\Phi^* = \Psi\Psi^* = n$$

Generalized Gross-Pitaevskii equation

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$$\Phi(t,\mathbf{x}) = \Psi(t,\mathbf{x})\exp(-i\omega_0 t)$$

$$-i\dot{\Psi} - \frac{1}{2\hat{m}}\nabla^2\Psi + \frac{\hat{\lambda}}{2\hat{m}}|\Psi|^2\Psi = 0,$$

$$\Phi\Phi^* = \Psi\Psi^* = n$$

Generalized Gross-Pitaevskii equation

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)\delta_{ij}dx^i dx^j$$

$$\nabla^0 = -\partial_0,$$
$$v_\mu = (-S, v),$$
$$\nabla^\mu = (-^{\cdot}, \nabla),$$
$$\hat{m} = \omega_0,$$
$$\Box = \nabla^2$$

Gross-Pitaevskii equation

$$\mathrm{i}\hbar\dot{\Psi} + \frac{\hbar^2}{2m}\nabla^2\Psi + g|\Psi|^2\Psi - m\phi\Psi = 0$$

The real Klein-Gordon equation

$$\Box^2\Phi - \frac{dV}{d\Phi} = 0$$

$$\kappa\Phi = \frac{1}{\sqrt{2}}\left(\Psi\, \mathrm{e}^{-\mathrm{i}\hat{m}ct} + \Psi^*\, \mathrm{e}^{\mathrm{i}\hat{m}ct}\right)$$

Klein-Gordon equation

$$\mathrm{i}\hbar\dot{\Psi} + \frac{\hbar^2}{2m}\Box^2\Psi + \frac{3\lambda}{2m}|\Psi|^2\Psi - m\phi\Psi + \frac{\lambda k_B^2 T^2}{8m}\Psi = 0$$

Gross-Pitaevskii equation

$$\mathrm{i}\hbar\dot{\Psi} + \frac{\hbar^2}{2m}\nabla^2\Psi + g|\Psi|^2\Psi - m\phi\Psi = 0$$

Hydrodynamical version of the Klein-Gordon equation

T. Matos, A. Avilez, T. Bernal and P-H. Chavanis. Energy Balance of a Bose gas in Curved Space-time. arXiv:1608.03945.

$$\Phi(t,\mathbf{x}) = \sqrt{n}\exp(i\omega) = \sqrt{n}\exp[i(S-\omega_0 t)]$$

$$\begin{aligned} \nabla^{\mu} J^{E}{}_{\mu} &= 0, \\ J^{E\mu} J^{E}{}_{\mu} + \frac{n^2}{\hat{m}^2}\left(\hat{m}^2 + \hat{\lambda} n - \frac{\Box\sqrt{n}}{\sqrt{n}}\right) &= 0 \end{aligned}$$

Bernoulli equation

$$J^{E}_{\mu} = \frac{n}{\hat{m}}\left(\nabla_{\mu}\omega + \hat{e}A_{\mu}\right)$$

Hydrodynamical version of the Klein-Gordon equation

T. Matos, A. Avilez, T. Bernal and P-H. Chavanis. Energy Balance of a Bose gas in Curved Space-time. arXiv:1608.03945.

$$\hat{m}v_\mu := \nabla_\mu S + \hat{e}A_\mu$$

$$\begin{aligned} -\frac{\omega_0}{\hat{m}}\left(\nabla^0 n + n\,\Box\, t\right) + \nabla^\mu(nv_\mu) &= 0, \\ \hat{m}^2 v_\mu v^\mu - 2\omega_0\hat{m}v^0 - \frac{\omega_0^2}{N^2} + \hat{m}^2 + \hat{\lambda}n - \frac{\Box\sqrt{n}}{\sqrt{n}} &= 0 \end{aligned}$$

Hydrodynamical version of the Klein-Gordon equation

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$$
\begin{aligned}
& v_\mu \nabla^\mu v_\alpha - \frac{\omega_0}{\hat{m}} \nabla^0 v_\alpha - \frac{\omega_0^2}{2\hat{m}^2} \nabla_\alpha \left(\frac{1}{N^2} \right) \\
+ \quad & \frac{\hat{\lambda}}{2\hat{m}^2} \nabla_\alpha n - \frac{1}{2\hat{m}^2} \nabla_\alpha \left(\frac{\Box\sqrt{n}}{\sqrt{n}} \right) \\
+ \quad & \frac{\hat{e}}{\hat{m}} \left[v_\mu (\nabla_\alpha A^\mu - \nabla^\mu A_\alpha) - \frac{\omega_0}{\hat{m}} (\nabla_\alpha A^0 - \nabla^0 A_\alpha) \right] = 0.
\end{aligned}
$$

Hydrodynamical version of the Klein-Gordon equation

T. Matos, A. Avilez, T. Bernal and P-H. Chavanis. Energy Balance of a Bose gas in Curved Space-time. arXiv:1608.03945.

$$F^E{}_\alpha := -\frac{\hat{e}}{\hat{m}}\left(v_\mu F_\alpha{}^\mu - \frac{\omega_0}{\hat{m}} F_\alpha{}^0\right)$$ Lorentz Force

$$F^G{}_\alpha := -\frac{\omega_0^2}{\hat{m}^2 N^3}\nabla_\alpha N$$ Gravitational Force

$$F^Q{}_\alpha := -\nabla_\alpha U^Q \qquad U^Q := -\frac{1}{2\hat{m}^2}\frac{\Box\sqrt{n}}{\sqrt{n}}$$ Quantum Force

$$F^n{}_\alpha := -\nabla_\alpha U^n \qquad U^n := \frac{\hat{\lambda} n}{2\hat{m}^2}$$ Hydro Force

$$-\frac{\omega_0}{\hat{m}}\nabla^0 v_\alpha + v_\mu \nabla^\mu v_\alpha = F^E{}_\alpha + F^G{}_\alpha + F^Q{}_\alpha + F^n{}_\alpha$$

The Newtonian Limit

P. H. Chavanis and T. Matos. arXiv:1606.07041

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)\delta_{ij}dx^i dx^j$$

$$\nabla^0 = -\partial_0,$$
$$N = 1 + 2\phi$$
$$v_\mu = (-S, v),$$
$$\nabla^\mu = (-^{\cdot}, \nabla),$$
$$\hat{m} = \omega_0,$$
$$\Box = \nabla^2$$

$$\begin{aligned}
\partial_0 n &+ \nabla\cdot(n\mathbf{v}) = 0,\\
\partial_0\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{m}\nabla U^Q &- \nabla\phi - \nabla U^n - \partial_0 P^E,\\
\partial_0 S + \frac{1}{2m}(\nabla S - e\mathbf{A})^2 = &- U^Q - U^n - U^G\\
&+ u^E - \nabla\cdot P^E,
\end{aligned}$$

The Gross-Pitaevskii equation

Tonatiuh Matos and Abril Suárez. Europhysics Letters EPL 96, (2011) 56005. Available at: arXiv:1110.3114.

Tonatiuh Matos, Elías Castellanos and Abril Suárez. Eur. Phys. J. C 77, (2017), 500. arXiv:1701.04894

$$\Psi = \sqrt{\hat{\rho}}\, \mathrm{e}^{\mathrm{i}S}$$

Klein-Gordon equation

$$\begin{aligned}
\mathrm{i}\hbar\dot{\Psi} \quad &+ \quad \frac{\hbar^2}{2m}\Box_E^2\Psi - \frac{\lambda}{2m\kappa^2}|\Psi|^2\Psi - m(\phi - 1)\Psi \\
&+ \quad i\frac{\hbar e}{2mc}(\boldsymbol{\nabla}\cdot\boldsymbol{A} - \dot{\varphi})\Psi - \frac{\hbar e^2}{2mc}(\boldsymbol{A}^2 - \varphi^2)\Psi \\
&+ \quad i\frac{\hbar e}{2mc}\left[\boldsymbol{A}\cdot\boldsymbol{\nabla}\Psi - \varphi(\dot{\Psi} - im\Psi)\right] \\
&- \quad \frac{\lambda\hbar}{8mc}k_B^2 T^2\Psi = 0,
\end{aligned}$$

The hydrodynamical version

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Klein-Gordon equation

$$\Psi = \sqrt{\hat{\rho}}\, \mathrm{e}^{\mathrm{i}S}$$

$$\dot{n} + \boldsymbol{\nabla}\cdot\left[n\frac{\hbar}{m}\left(\boldsymbol{\nabla S} - \frac{e}{\hbar}\boldsymbol{A}\right)\right] - \left[n\frac{\hbar}{m}\left(\dot{S} - \frac{e}{\hbar}\varphi\right)\right]^{\cdot} = 0,$$

$$\begin{aligned}\frac{\hbar}{m}\left(\dot{S} - \frac{e}{\hbar}\varphi\right) &+ \frac{\lambda}{2m^2\kappa^2}n + (\phi - 1) + \frac{\lambda}{8m^2}k_B^2T^2 \\ &+ \frac{1}{2}[\left[\frac{\hbar}{m}\left(\boldsymbol{\nabla}S - \frac{e}{\hbar}\boldsymbol{A}\right)\right]^2 \\ &- \left[\frac{\hbar}{m}\left(\dot{S} - \frac{e}{\hbar}\varphi\right)\right]^2] \\ &+ \frac{\hbar^2}{2m^2}\left(\frac{\Box^2\sqrt{n}}{\sqrt{n}}\right) = 0.\end{aligned}$$

The hydrodynamical version

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$$n\dot{\boldsymbol{v}} + n(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = n\boldsymbol{F}_E + n\boldsymbol{F}_\phi - \boldsymbol{\nabla} p + n\boldsymbol{F}_Q + \boldsymbol{\nabla}\sigma,$$

$$\boldsymbol{F}_E = \frac{e}{m}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

$$\boldsymbol{F}_\phi = -\boldsymbol{\nabla}\phi$$

$$p = \omega n^2$$

$$U_Q = \frac{\hbar^2}{m^2}\left(\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right)$$

$$\boldsymbol{F}_Q = -\boldsymbol{\nabla} U_Q$$

The viscosity

Tonatiuh Matos and Abril Suárez. Europhysics Letters EPL 96, (2011) 56005. Available at: arXiv:1110.3114.
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$$\begin{aligned}\boldsymbol{\nabla}\sigma &= \frac{\hbar}{2m}n\boldsymbol{\nabla}\left(\dot{S}-e\varphi\right)^2-\frac{1}{4}\frac{\lambda}{m^2}k_B^2 nT\boldsymbol{\nabla}T\\ &- \zeta\boldsymbol{\nabla}\dot{(\ln n)}+\frac{\hbar^2 n}{2m^2}\boldsymbol{\nabla}\left(\frac{\ddot{n}}{n}\right),\end{aligned}$$

$$\dot{n}+\boldsymbol{\nabla}\cdot(n\boldsymbol{v}) = 0$$

The viscosity

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$$\boldsymbol{\nabla}\sigma = -\frac{1}{4}\frac{\lambda}{m}k_B^2 nT\boldsymbol{\nabla}T - \zeta\left[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + \boldsymbol{\nabla}[\boldsymbol{\nabla}(\ln n)\cdot\boldsymbol{v}]\right],$$

The Newtonian Limit

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P. H. Chavanis and T. Matos. arXiv:1606.07041

$$
\begin{aligned}
\dot{n} + \boldsymbol{\nabla} \cdot (n\boldsymbol{v}) &= 0, \\
n\dot{\boldsymbol{v}} + n(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} &= n\boldsymbol{F}_E + n\boldsymbol{F}_\phi - \boldsymbol{\nabla} p + n\boldsymbol{F}_Q + \boldsymbol{\nabla}\sigma.
\end{aligned}
$$

$$\boldsymbol{F}_E = \frac{e}{m}\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$$

$$F^{\mu\nu}{}_{,\nu} = -\frac{1}{\kappa^2} j^\mu$$

$$\boldsymbol{\nabla}\sigma = \frac{1}{4}\frac{\lambda}{m} k_B^2 \hat{\rho} T \boldsymbol{\nabla} T - \zeta\left[\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) + \boldsymbol{\nabla}[\boldsymbol{\nabla}(\ln\hat{\rho}) \cdot \boldsymbol{v}]\right]$$

$$\zeta = -\frac{\hbar^2}{4m^2} \boldsymbol{\nabla} \cdot (\hat{\rho}\boldsymbol{v})$$

Structure Formation

Abril Suarez and TM MNRAS 311, (2011), 87

$$\delta\ddot{\Phi} + 3H\delta\dot{\Phi} - \frac{1}{a^2}\nabla^2\delta\Phi + V_{,\Phi\Phi}\,\delta\Phi + 2V_{,\Phi}\,\phi = 0$$

$$\delta\Phi = \sqrt{\hat{\rho}}\,\cos\left(\frac{mc^2\,t}{\hbar} + S\right)$$

$$\vec{v} \equiv \frac{\hbar}{m}\nabla S$$

Structure Formation

Abril Suarez and TM MNRAS 311, (2011), 87

$$\vec{v} \equiv \frac{\hbar}{m}\nabla S$$

$$\frac{\partial\hat{\rho}}{\partial t} + \frac{1}{a^2}\nabla\cdot(\hat{\rho}\vec{v}) + 3H\hat{\rho} = 0$$

$$- \frac{\hbar}{m}\hat{\rho}\left(\ddot{S} + 3H\dot{S}\right) - \frac{\hbar}{m}\dot{\hat{\rho}}\dot{S} = 0$$

$$\frac{\partial\vec{v}}{\partial t} + \frac{1}{a^2}\vec{v}\nabla\cdot\vec{v} + \nabla\phi + \frac{\hbar^2}{2m^2}\nabla\left(\frac{\Box\sqrt{\hat{\rho}}}{\sqrt{\hat{\rho}}}\right) = 0,$$

$$- \frac{\hbar}{m}\dot{S}\,\frac{\partial\vec{v}}{\partial t} = 0,$$

Structure Formation

Abril Suarez and TM MNRAS 311, (2011), 87

$$\hat{\rho} = \hat{\rho}(t) \qquad S = S(t) \qquad \vec{v} \equiv \frac{\hbar}{m}\nabla S$$

$$\frac{\partial \hat{\rho}}{\partial t} + 3H\hat{\rho} = 0$$

$$\frac{\partial \hat{\rho}}{\partial t} + \frac{1}{a^2}\nabla \cdot (\hat{\rho}\vec{v}) + 3H\hat{\rho} = 0$$

$$\hat{\rho} \sim \frac{1}{a^3}$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{a^2}\vec{v}\nabla \cdot \vec{v} + \nabla\phi + \frac{\hbar^2}{2m^2}\nabla\left(\frac{\Box\sqrt{\hat{\rho}}}{\sqrt{\hat{\rho}}}\right) = 0,$$

Structure Formation

Abril Suarez and TM MNRAS 311, (2011), 87

$$\vec{v} \equiv \frac{\hbar}{m}\nabla S$$

$$\frac{\partial \hat{\rho}}{\partial t} + \frac{1}{a^2}\nabla \cdot (\hat{\rho}\vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{a^2}\vec{v}\nabla \cdot \vec{v} + \nabla\phi + \frac{\hbar^2}{2m^2}\nabla\left(\frac{\Box\sqrt{\hat{\rho}}}{\sqrt{\hat{\rho}}}\right) = 0,$$

Structure Formation

Abril Suarez and TM MNRAS 311, (2011), 87

$$\begin{aligned}\hat{\rho} &= \hat{\rho}_0 + \rho_1(t)\exp(i\vec{k}\cdot\vec{x}/a)\\ \vec{v} &= \vec{v}_0 + \vec{v}_1(t)\exp(i\vec{k}\cdot\vec{x}/a)\\ \phi &= \phi_0 + \phi_1(t)\exp(i\vec{k}\cdot\vec{x}/a)\end{aligned}$$

$$\vec{v}_1 = \lambda\vec{k} + \vec{v}_2$$

$$\begin{aligned}\frac{\partial\rho_1}{\partial t} + 3H\rho_1 + i\frac{\hat{\rho}_0}{a^2}k^2\lambda &= 0\\ \frac{\partial\lambda}{\partial t} + i\left(\frac{v_q^2}{\hat{\rho}_0} - 4\pi G\frac{a^2}{k^2}\right)\rho_1 &= 0,\end{aligned}$$

$$\delta = \frac{\rho_1}{\hat{\rho}}$$

SFDM

$$\frac{d^2\delta}{dt^2} + 2\,H\frac{d\delta}{dt} + \left(v_q^2\frac{k^2}{a^2} - 4\pi G\hat{\rho}_0\right)\delta = 0, \qquad v_q^2 = \frac{\hbar^2 k^2}{4a^2m^2}$$

CDM

$$\frac{d^2\delta}{dt^2} + 2\,H\frac{d\delta}{dt} + \left(v_s^2\frac{k^2}{a^2} - 4\pi G\hat{\rho}_0\right)\delta = 0,$$

The Balance equations

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$$\begin{aligned}
\dot{(n\psi_E)} &= \dot{n}\psi_i + n\dot{\psi}_i \\
&= -\boldsymbol{\nabla}\cdot(n\boldsymbol{v}\psi_i) + n\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi_i + n\dot{\psi}_i \\
&= -\boldsymbol{\nabla}\cdot(n\boldsymbol{v}\psi_i) - n\boldsymbol{v}\cdot\boldsymbol{F}_i + n\dot{\psi}_i - n\boldsymbol{v}\cdot\boldsymbol{j}_i
\end{aligned}$$

$$\dot{n} + \boldsymbol{\nabla}(n\boldsymbol{v}) = 0$$

$$\psi_i = \{\phi, u, E\}$$

$$\dot{(n\psi_i)} + \boldsymbol{\nabla}\cdot(n\boldsymbol{v}\psi_i) + \boldsymbol{J}_i = n\dot{\psi}_i - n\boldsymbol{v}\cdot\boldsymbol{F}_i$$

The Balance equations

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$$n\dot{\boldsymbol{v}} + n(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = n\boldsymbol{F}_E + n\boldsymbol{F}_\phi - \boldsymbol{\nabla}p + n\boldsymbol{F}_Q + \boldsymbol{\nabla}\sigma,$$

$$\epsilon = \frac{1}{2}nv^2 + n\phi + nu + nU_Q + \psi_E \qquad \psi_E = \frac{e}{m}(\varphi - \boldsymbol{v}\cdot\boldsymbol{A})$$

$$(nu)^{\cdot} + \boldsymbol{\nabla}\cdot(n\boldsymbol{v}u + \boldsymbol{J}_q + \boldsymbol{J}_B - p\boldsymbol{v} - \boldsymbol{J}_\rho) = -p\boldsymbol{\nabla}\cdot\boldsymbol{v}$$

such that, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_q = \boldsymbol{v}\cdot(\boldsymbol{\nabla}\sigma)$, $\quad \boldsymbol{J}_\rho = n\boldsymbol{v}_\rho \quad \boldsymbol{v}_\rho = \frac{\hbar^2}{4m^2}(\boldsymbol{\nabla}\ln n)^{\cdot}$,

and $\boldsymbol{\nabla}\cdot\boldsymbol{J}_B = \boldsymbol{v}\cdot(n\boldsymbol{j}_B)$.

The Balance equations

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$$(nu)^{\cdot} + \boldsymbol{\nabla}\cdot(n\boldsymbol{v}u + \boldsymbol{J}_q + \boldsymbol{J}_B - p\boldsymbol{v} - \boldsymbol{J}_\rho) = -p\boldsymbol{\nabla}\cdot\boldsymbol{v}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int nu\,\mathrm{d}V \quad &+ \quad \oint(\boldsymbol{J}_q + \boldsymbol{J}_B + p\boldsymbol{v})\cdot\boldsymbol{n}\,\mathrm{d}S - \oint \boldsymbol{J}_\rho\cdot\boldsymbol{n}\,\mathrm{d}S \\ &= \quad -p\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}V.\end{aligned}$$

$$\mathrm{d}U = \hat{\mathrm{d}}Q + \hat{\mathrm{d}}Q_B + \hat{\mathrm{d}}A_Q - p\mathrm{d}V$$

$$\frac{\hat{\mathrm{d}}A_Q}{\mathrm{d}t} = \oint n\boldsymbol{v}_\rho\cdot\boldsymbol{n}\,\mathrm{d}S$$

The Thermodynamics

$$\mathrm{d}U = \hat{\mathrm{d}}Q + \hat{\mathrm{d}}Q_B + \hat{\mathrm{d}}A_Q - p\mathrm{d}V$$

$$\begin{aligned}
\frac{\hat{\mathrm{d}}Q_B}{\mathrm{d}t} &= \int \boldsymbol{\nabla} \cdot \boldsymbol{J}_B \, dV = \int \boldsymbol{v} \cdot (n\boldsymbol{j}_B) \, dV \\
&= \frac{m}{e} \int n \left(\frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{A} + (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{v} \right) \cdot \boldsymbol{v} \, dV
\end{aligned}$$

$$\frac{\hat{\mathrm{d}}A_Q}{\mathrm{d}t} = \oint n v_\rho \cdot n \, \mathrm{d}S$$

$$\Box \boldsymbol{A} = -\frac{1}{\kappa^2} \boldsymbol{j}$$

$$\Box \varphi = -\frac{1}{\kappa^2} j$$

The Balance equations

T. Matos, A. Avilez, T. Bernal and P-H. Chavanis. Energy Balance of a Bose gas in Curved Space-time. arXiv:1608.03945.

$$
\begin{aligned}
- &\quad \frac{\omega_0}{\hat{m}}\nabla^0(n\mathcal{U}_i) + \frac{\omega_0}{\hat{m}}(\nabla^0 n)\mathcal{U}_i \\
+ &\quad \frac{\omega_0}{\hat{m}} n(\nabla^0 \mathcal{U}_i) = 0,
\end{aligned}
$$

$$\{\mathcal{U}_i\}_{i=1,2,3} = \{U^n, U^Q, U^G\}$$

$$
\begin{aligned}
-\frac{\omega_0}{\hat{m}}\nabla^0(n\mathcal{U}_i) \quad &+ \quad \nabla^\mu(v_\mu n\mathcal{U}_i) - n v_\mu \nabla^\mu \mathcal{U}_i \\
+\frac{\omega_0}{\hat{m}} n\nabla^0\mathcal{U}_i \quad &+ \quad \frac{\omega_0}{\hat{m}} n\mathcal{U}_i \Box t = 0,
\end{aligned}
$$

$$-\frac{\omega_0}{\hat{m}}\left(\nabla^0 n + n\,\Box\, t\right) + \nabla^\mu(n v_\mu) = 0$$

The Balance equations

T. Matos, A. Avilez, T. Bernal and P-H. Chavanis. Energy Balance of a Bose gas in Curved Space-time. arXiv:1608.03945.

$$\begin{aligned} - & \frac{\omega_0}{\hat{m}} \nabla^0 (n \mathcal{U}_i) + \frac{\omega_0}{\hat{m}} (\nabla^0 n) \mathcal{U}_i \\ + & \frac{\omega_0}{\hat{m}} n (\nabla^0 \mathcal{U}_i) = 0, \end{aligned}$$

$$\{\mathcal{U}_i\}_{i=1,2,3} = \{U^n, U^Q, U^G\}$$

$$\begin{aligned} -\frac{\omega_0}{\hat{m}} \nabla^0 (n \mathcal{U}_i) \quad & + \quad \nabla^\mu (v_\mu n \mathcal{U}_i + \mathcal{J}_i) + n v_\mu \mathcal{F}_i^\mu \\ + \frac{\omega_0}{\hat{m}} n \nabla^0 \mathcal{U}_i \quad & + \quad \frac{\omega_0}{\hat{m}} n \mathcal{U}_i \Box t = 0 \end{aligned}$$

$$-\frac{\omega_0}{\hat{m}} \left(\nabla^0 n + n \Box t \right) + \nabla^\mu (n v_\mu) = 0$$

The Balance equations

T. Matos, A. Avilez, T. Bernal and P-H. Chavanis. Energy Balance of a Bose gas in Curved Space-time. arXiv:1608.03945.

$$
\begin{aligned}
- \quad & \frac{\omega_0}{\hat{m}}\nabla^0(n\mathcal{U}^s) + \nabla^\mu(nv_\mu\mathcal{U}^s) + nv_\mu J^{E\mu} + nv_0\nabla^0 U^E \\
+ \quad & \nabla^\mu\left({J^Q}_\mu + pv_\mu\right) - p\nabla^\mu v_\mu + \frac{\omega_0}{\hat{m}} n\nabla^0 U^G \\
+ \quad & \frac{\omega_0}{\hat{m}} n(U^G + U^Q)\Box t = 0
\end{aligned}
$$

$$\mathcal{U}^s := K + U^G + U^Q + U^n$$

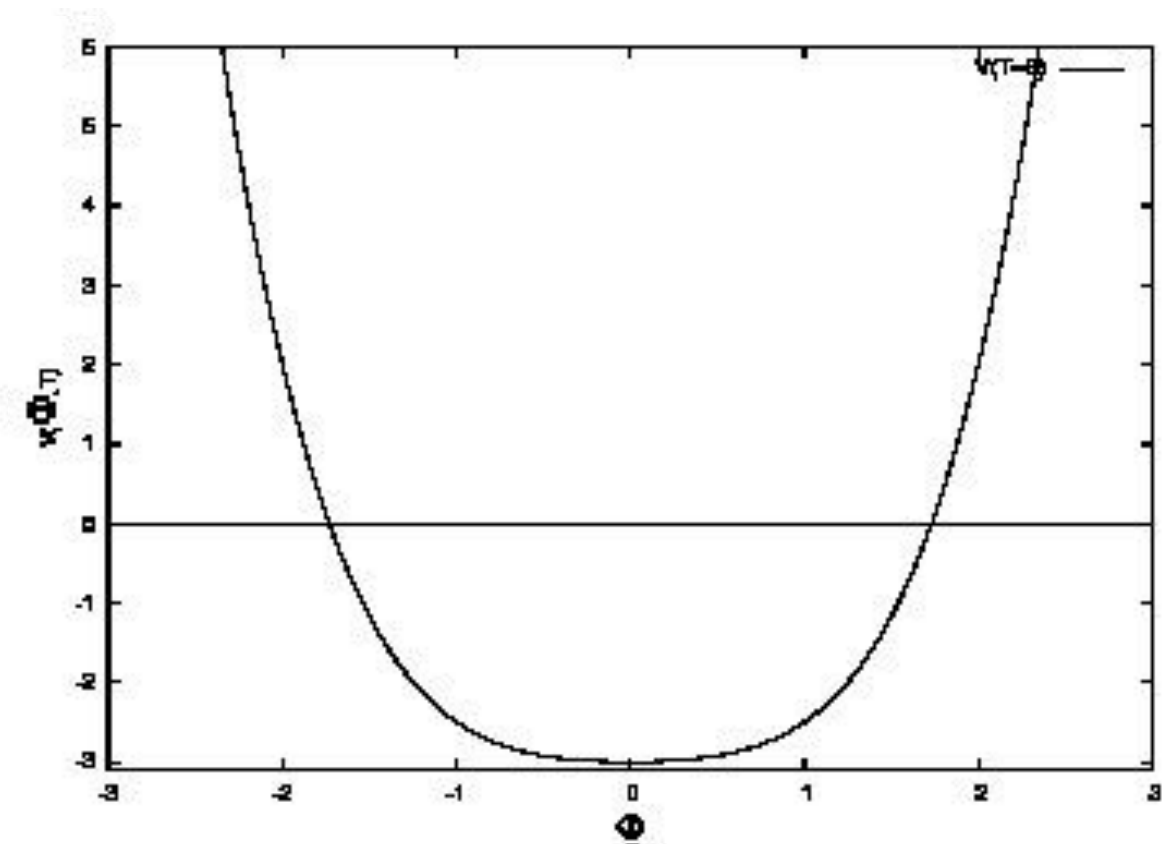
$$K := \frac{1}{2}v_\alpha v^\alpha \qquad U^G = -\frac{\omega_0^2}{4\hat{m}^2N^2}$$

$$U^Q := -\frac{1}{2\hat{m}^2}\frac{\Box\sqrt{n}}{\sqrt{n}} \qquad U^n := \frac{\hat{\lambda}n}{2\hat{m}^2}$$

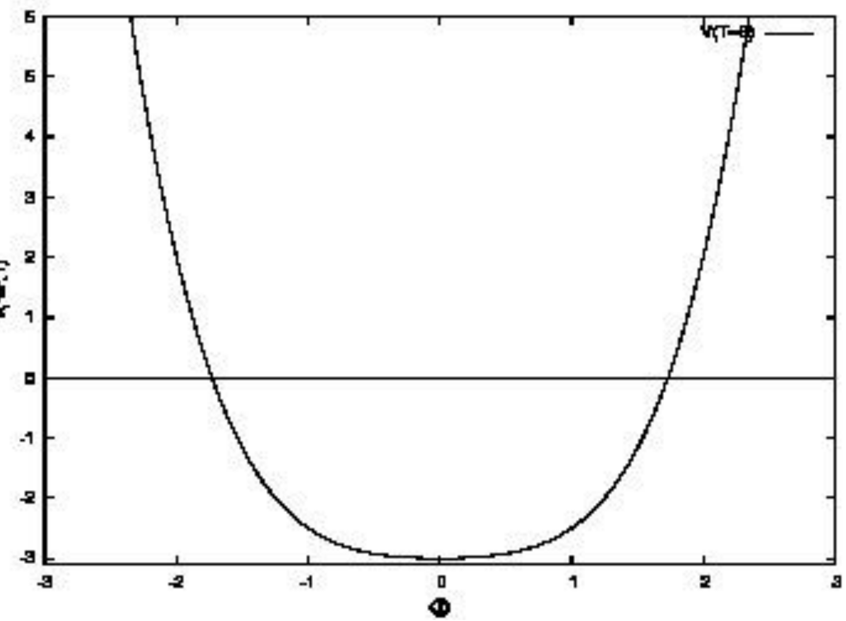
The Phase Transition

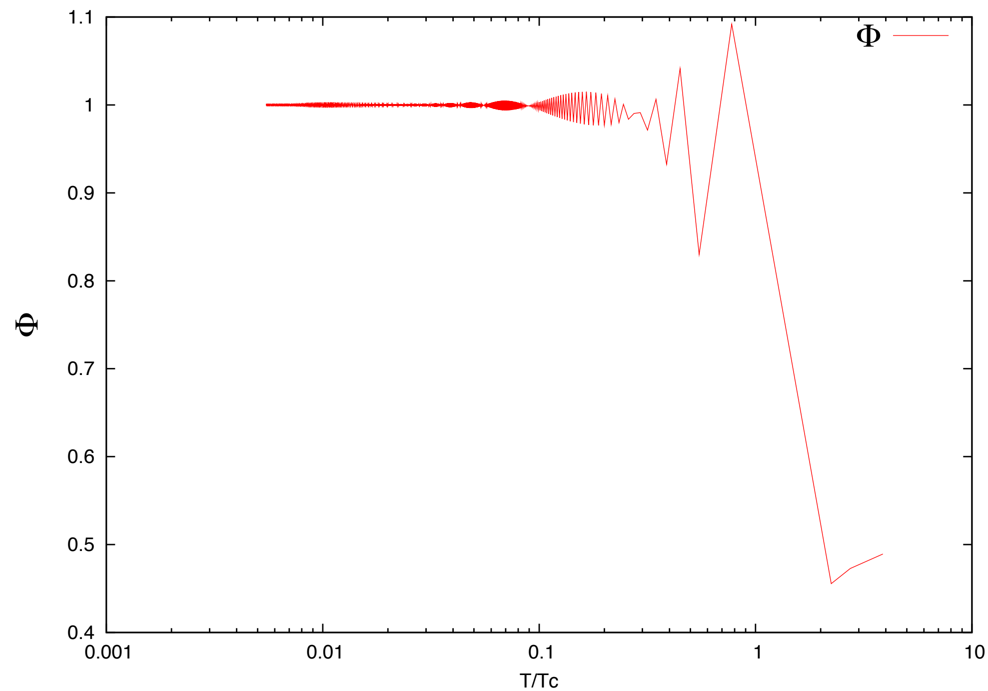
$$T_c = \frac{2m_{eff}}{\sqrt{\lambda}} \qquad m_{eff}c = \sqrt{m^2c^2 + e^2A^2} \qquad \Phi_{min} = \pm\frac{k_B}{2}\sqrt{T_c^2 - T^2}$$

$$T \lessgtr T_c$$



The Phase Transition





$$\Phi = \sqrt{\hat{\rho}}\,\cos\left(\frac{mc^2\,t}{\hbar} + S\right)$$

The Phase Transition

$$S \rightarrow S(t)$$

$$\Box \boldsymbol{A} = -\frac{1}{\kappa^2}\boldsymbol{j}$$

$$\Box \boldsymbol{A} = -\frac{2ne^2}{\kappa^2 \hbar}\boldsymbol{A}$$

$$\vec{v} \equiv \frac{\hbar}{m}\nabla S$$

$$\Box \varphi = -\frac{1}{\kappa^2} j$$

$$\boldsymbol{\nabla}\sigma = \frac{1}{4}\frac{\lambda}{m}k_B^2 \hat{\rho} T \boldsymbol{\nabla} T - \zeta \left[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + \boldsymbol{\nabla}[\boldsymbol{\nabla}(\ln\hat{\rho})\cdot\boldsymbol{v}]\right]$$

$$\zeta = -\frac{\hbar^2}{4m^2}\boldsymbol{\nabla}\cdot(\hat{\rho}\boldsymbol{v})$$

$$\begin{aligned} \boldsymbol{j} &= 2en(\nabla S + \frac{e}{\hbar}\boldsymbol{A}) \\ j &= 2en(\dot{S} + \frac{e}{\hbar}\varphi) \\ j_\mu &= (\boldsymbol{j}, j) \end{aligned}$$

$$\boldsymbol{v} \equiv \frac{\hbar}{m}(\boldsymbol{\nabla} S + e\boldsymbol{A})$$

Axial Symmetry and Stationarity

Luis Padilla-Albores, A. Avilez, T. Bernal, T. Matos, and P-H. Chavanis. In preparation

$$ds^2 = - f(dt - \Omega d\phi)^2 + \frac{1}{f}[K dl^2 + (l^2 - 2l_1 l + l_0^2)(K d\theta^2 + \sin^2(\theta) d\varphi^2)]$$

$$\frac{f}{K(l^2 - 2l_1 l + l_0^2)} \Big[((l^2 - 2l_1 l + l_0^2)L_{,l})_{,l} + \left(\frac{j^2}{\sin^2(\theta)} - k(k+1) - \frac{K\left((j+\Omega\omega) + \hat{e}(A_\varphi + \Omega A_t)\right)^2}{\sin^2(\theta)}\right) L\Big] + \left(\frac{1}{f}(\omega + \hat{e}A_t)^2 - (m^2 + \lambda n)\right) L = 0$$

$$\Phi = \Psi \exp(-i\omega_0 t) = L(l) Y_k^j(\theta, \varphi) T(t) \exp(-i\omega_0 t)$$

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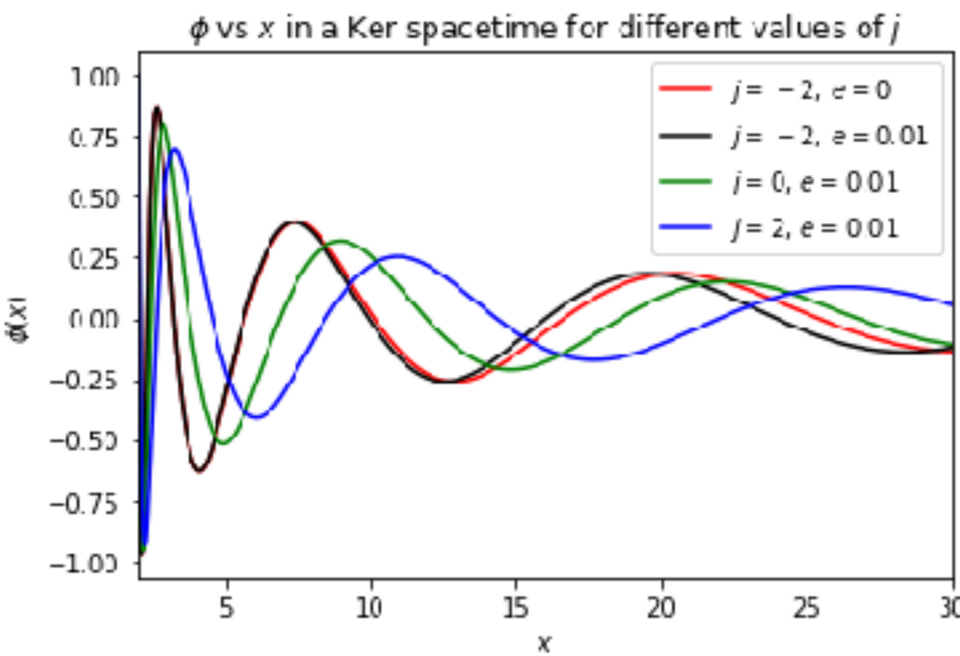

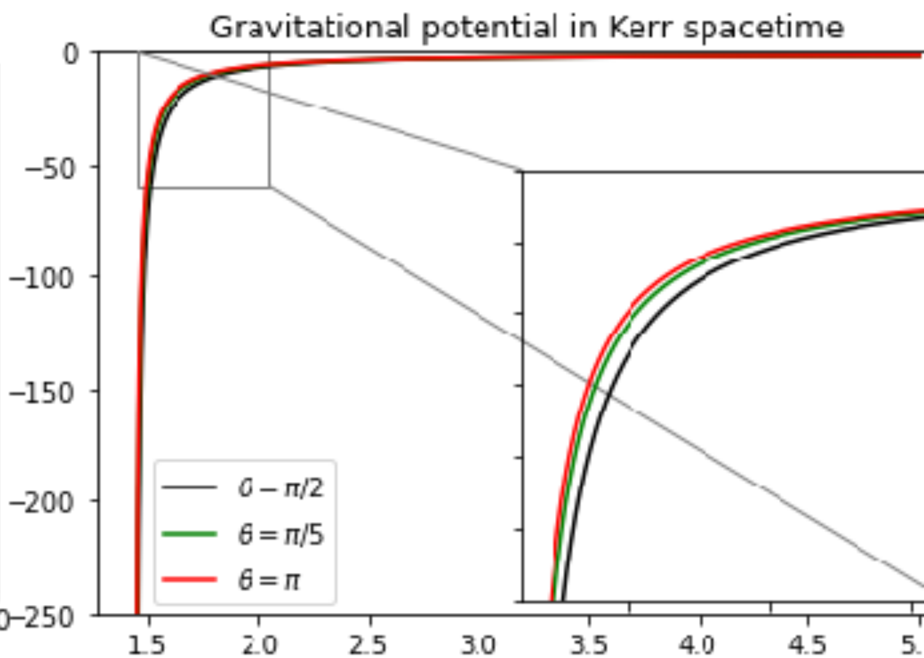

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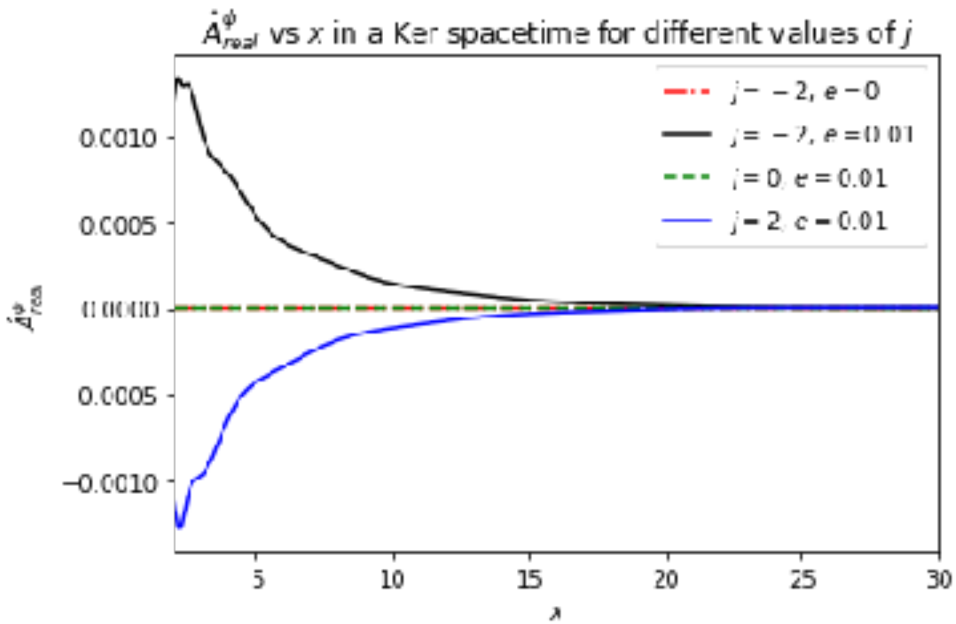

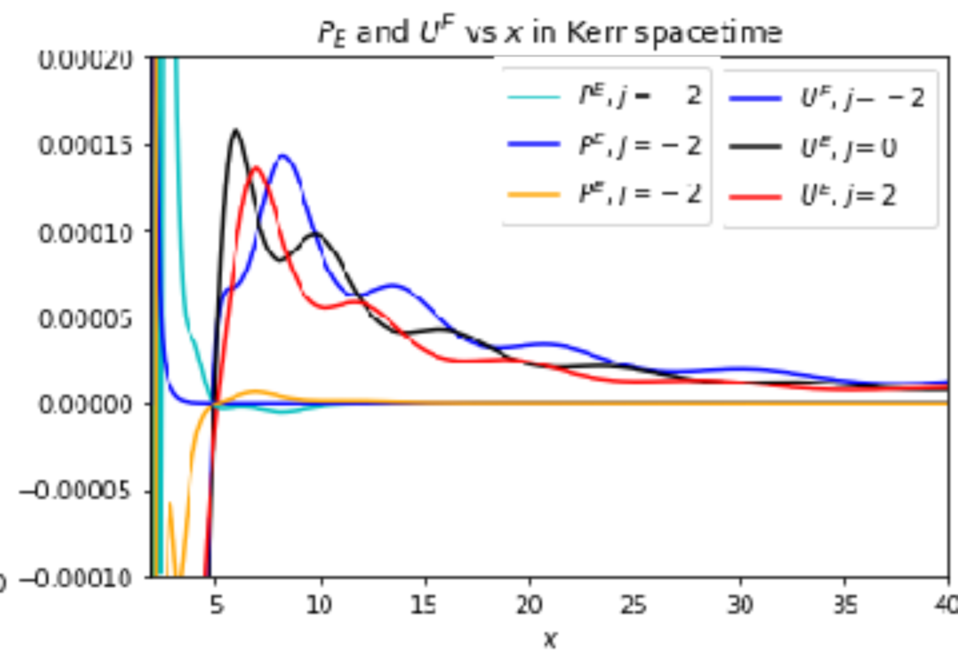

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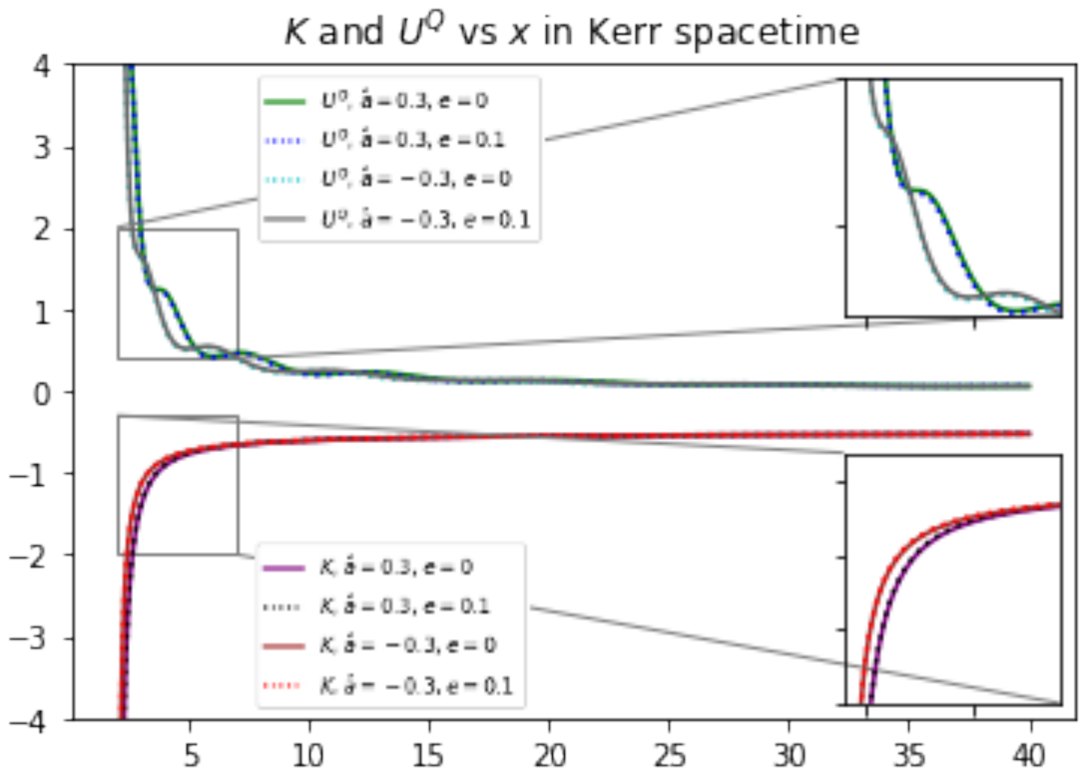

Conclusion

- By performing a 3+1 foliation of the space-time, we have shown that it is possible to split the total energy of the system into different contributions, specifically the quantum and electromagnetic parts, and a term due to the gravitational field strength. The main result of this letter is the energy balance equation for the boson-gas playing the role of the first law of thermodynamics. It is worth to remark that this result is thoroughly general and it has not been derived before.